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1 Feature Modelling Derivations

1.1 Gaussian Maximisation Lemma

Lemma 1. *Given two Gaussian distributions*

$$p_h(\cdot) = \mathcal{N}(\cdot; \mu, \Sigma_h), \qquad p_x(\cdot) = \mathcal{N}(\cdot; 0, \Sigma_x), \tag{1}$$

the following identity holds:

$$\max_x \left[\log p_h(y-x) + \log p_x(x)\right] = \log \mathcal{N}(y; \mu, \Sigma_h + \Sigma_x) + C. \tag{2}$$

***Proof**: Expanding the Gaussian densities yields*

$$\log p_h(y-x) + \log p_x(x) = -\frac{1}{2}\left[(y-x-\mu)^T \Sigma_h^{-1}(y-x-\mu) + x^T \Sigma_x^{-1} x\right] + C' \tag{3}$$

$$= -\frac{1}{2}\underbrace{\left[(v-x)^T \Lambda_h (v-x) + x^T \Lambda_x x\right]}_{R(x)} + C', \tag{4}$$

where $\Lambda_h = \Sigma_h^{-1}$, $\Lambda_x = \Sigma_x^{-1}$, and $v = y - \mu$. Setting the derivative to zero:

$$\frac{dR}{dx} = -2\Lambda_h(v-x) + 2\Lambda_x x = 0 \implies x^* = (\Lambda_h + \Lambda_x)^{-1}\Lambda_h v. \tag{5}$$

The Hessian $2(\Lambda_h + \Lambda_x) \succ 0$ confirms x^* is a global minimum. At the optimum, stationarity gives $\Lambda_h(v - x^*) = \Lambda_x x^*$, so

$$R(x^*) = (v-x^*)^T \Lambda_h (v-x^*) + {x^*}^T \Lambda_x x^* \tag{6}$$

$$= (v-x^*)^T \Lambda_x x^* + {x^*}^T \Lambda_x x^* \tag{7}$$

$$= v^T \Lambda_x x^*. \tag{8}$$

Finally, substituting in the expression for x^*, we obtain

$$R(x^*) = v^T \Lambda_x (\Lambda_h + \Lambda_x)^{-1} \Lambda_h v \tag{9}$$

$$= v^T (\Sigma_h + \Sigma_x)^{-1} v \tag{10}$$

$$= (y-\mu)^T (\Sigma_h + \Sigma_x)^{-1} (y-\mu), \tag{11}$$

where the penultimate step uses $(A^{-1} + B^{-1})^{-1} = A(A+B)^{-1}B$. Substituting $R(x^*)$ into Equation 4 completes the proof.

1.2 Normalising Flow Energy Derivation

The inverse corruption function is given by $h = f^{-1}(y, x) = g(w - x)$ where $w = g^{-1}(y)$. The healthy energy is obtained by substituting h into the normalising

flow likelihood:

$$E_{\text{healthy}}(h, x) = -\log p_z(g^{-1}(h)) - \log \left|\det J_{g^{-1}}(h)\right| \tag{12}$$
$$= -\log p_z(w - x) - \log \left|\det J_{g^{-1}}(g(w - x))\right| \tag{13}$$
$$= -\log p_z(w - x) + \log \left|\det J_g(w - x)\right|, \tag{14}$$

where we used the inverse function theorem in the last line. Using the alternate expression for the corruption Jacobian in terms of its inverse and applying the chain rule, the volume energy is

$$E_{\text{volume}}(h, x) = -\log \left|\det J_{f^{-1}}(y, x)\right| \tag{15}$$
$$= -\log \left|\det J_g(w - x)\right| - \log \left|\det J_{g^{-1}}(y)\right|. \tag{16}$$

Finally, the anomaly energy is simply $E_{\text{anomaly}}(x) = -\log p_x(x)$. Summing the three terms yields the desired expression:

$$E(x, y) = -\log p_z(w - x) - \log p_x(x) - \log \left|\det J_{g^{-1}}(y)\right|. \tag{17}$$