1 Appendix - Figures

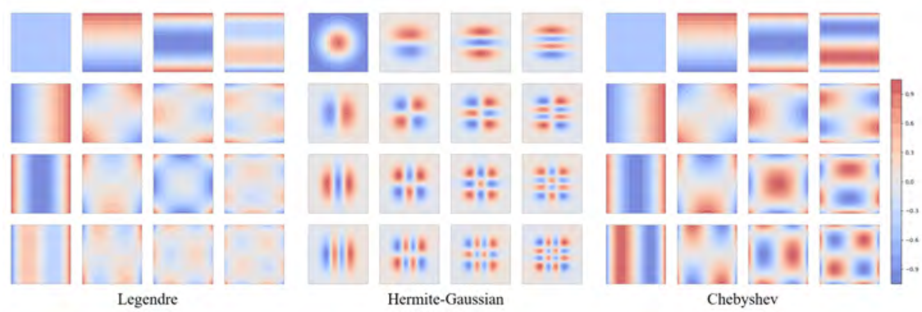

Fig. 1. Illustration of 2D 4th order Legendre, Hermite-Gaussian functions, and Chebyshev polynomials.

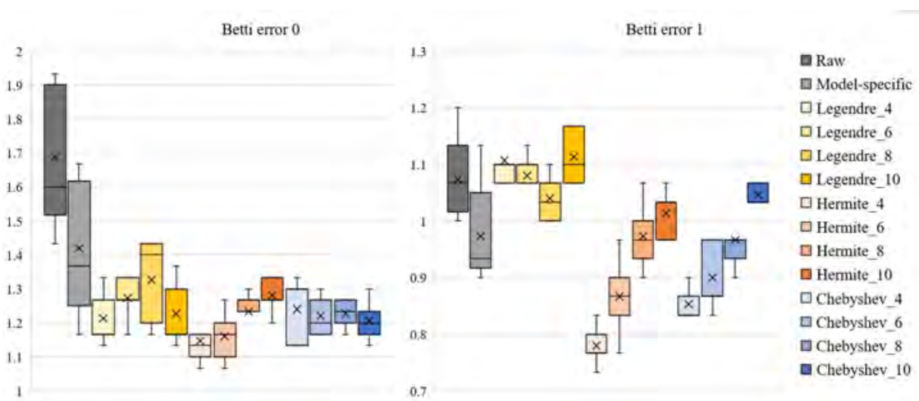

Fig. 2. Ablation study: Comparing the topological performance of our model trained using synthetic segmentations generated from various polynomials: Legendre, Hermite-Gaussian, and Chebyshev polynomials with order 4, 6, 8, and 10.

2 Appendix - Relative Mathematical Proof

Theorem 1 (StoneâĂŞWeierstrass theorem). *Suppose X is a compact Hausdorff space and A is a subalgebra of $\mathrm{C}(X, \mathbb{R})$ which contains a non-zero constant function. Then A is dense in $\mathrm{C}(X, \mathbb{R})$ if and only if it separates points.*

Corollary 1 (Approximation on tensor product of polynomials). *For a compact domain $D \subset \mathbb{R}^n$ and $C(D)$, the space of continuous functions on D with supremum norm, tensor products of orthogonal polynomials on D form a dense subset in $C(D)$, enabling arbitrary close approximations of any $f \in C(D)$.*

Proof. Let $D \subseteq \mathbb{R}^n$ be a compact domain with $C(D)$ representing the space of continuous functions on D endowed with the supremum norm. Define $\{P_k^{(i)}\}_{k\in\mathbb{N}}$ as orthogonal polynomials over each coordinate interval of D, for $i = 1, \ldots, n$. The algebra A, generated by $\bigotimes_{i=1}^n P_{k_i}^{(i)}$, separates points and includes 1, meeting Stone-Weierstrass prerequisites.

For any $f \in C(D)$ and $\epsilon > 0$, there exists $g \in A$, $g = \sum c_{k_1,\ldots,k_n} \bigotimes_{i=1}^n P_{k_i}^{(i)}$, such that $||f - g||_\infty < \epsilon$. Hence, tensor products of orthogonal polynomials are dense in $C(D)$, ensuring approximation of any $f \in C(D)$.

Proposition 1. *The Chebyshev Polynomial $T_n(x)$ has following properties*

1. *n simple zeros in $[-1, 1]$ at $\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right)$ for integers k*
2. *has bounded derivative $|T_n'(x)| \leq n^2$ for $|x| \leq 1$ and $|T_n'(\pm 1)| = n^2$*
3. *$T_n(x)$ and $T_n'(x)$ share no common zeros.*
4. *T_n has $n - 1$ critical points when $n \geq 1$*

Proof. 1. Roots can be verified by plugging into T_n
2. On $[0, 1]$ let $x = \cos(\theta)$, $T_n(x) = \cos(n\theta)$

$$\frac{d}{dx}T_n(x) = \frac{\frac{d}{d\theta}T_n(\cos(\theta))}{\frac{d}{d\theta}\cos(\theta)} = \frac{n\sin(n\theta)}{\sin(\theta)} \tag{1}$$

For $|x| \leq 1, |T_n'(x)| \leq n^2$, since $|\sin(n\theta)| \leq n|\sin(\theta)|$. For $x = \pm 1, |T_n'(\pm)| = n^2$, by L'HÃťpital rule in the limit: $\theta \to 0$ and $\theta \to \pi$ we find $|T_n'(\pm 1)| = n^2$. 3. By eq.1 $\sin(n\theta)$ shares no common zeros as $\cos(n\theta)$. 4. Can be checked by eq.1.

Definition 1 (Morse Function). *A map $f : M \to \mathbb{R}$ is a Morse function if its critical points are nondegenerate.*

Proposition 2. *Let $F : [0, 1]^d \to [0, 1]$ denote a d-dimensional probability density function, and consider a d-dimensional Chebyshev Polynomial basis $T_i(\boldsymbol{x}) = T_{i_1}(x_1) \cdot T_{i_2}(x_2) \cdot \ldots \cdot T_{i_d}(x_d)$ for $\boldsymbol{x} = (x_1, \ldots, x_d)$. Suppose $G = m \cdot F$ represents the new density function after applying a normalized mask $m = \sum a_{\boldsymbol{i}} T_{\boldsymbol{i}}(\boldsymbol{x})$ on F. Then critical points $\boldsymbol{x}_G$, satisfying $\nabla G(\boldsymbol{x}_G) = 0$, are dependent by the original critical points $\boldsymbol{x}_F$, $\nabla F(\boldsymbol{x}_F) = 0$, according to the relationship:*

$$\nabla G = m\nabla F + F\nabla m = 0$$

Also, the Hessian matrix of G at $c \in [0, 1]^d$ is well-defined and is given by the relation with $H_c(F)$:

$$\begin{aligned} H_c(G)^T &= \nabla_c (\nabla_c G) = \nabla_c ((\nabla_c m) F + m (\nabla_c F)) \\ &= H_c(F)^T m(c) + \nabla_c F^T \nabla_c m + \nabla_c m^T \nabla_c F + F(c) H_c(m)^T. \end{aligned}$$

The index is given by $\mathrm{index}(c) = \# \left\{\lambda_i < 0 \mid \lambda_i \in \text{ eigenvalues } \left(H_c(G)^T\right)\right\}$